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Modeling economic returns with the skew-t distribution

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ABSTRACT
There exists considerable interest in the probabilistic modeling of economic variables to compute financial estimates with the least possible uncertainty. Most studies are based on symmetric probability distributions, specifically on the normal distribution; however, empirical evidence indicates that the probability distributions of variables such as economic returns present a certain degree of skewness. Therefore, the main issue in this paper is to propose a probability distribution which provides a good fit to economic returns. In particular, this article focuses on the analysis of the economic returns of the Mexican $S\&P/BMV\ IPC$ Index, and it is found that the skew-t distribution might be a plausible model for this variable. A new probability property of the skew-t family of distributions is introduced here, and it is used to propose a procedure for testing the skew-t distribution hypothesis. The results of a Monte Carlo simulation study conducted in order to study the power properties of this test are also presented.

KEYWORDS
Economic returns; skew-t distribution; Anderson-Darling test; Monte Carlo method; parametric bootstrap.

1. Introduction

The parametric statistical inference relies on the assumption that the data constitute a random sample from a population that has a given probability distribution, which usually depends on unknown parameters. In several cases, parametric inferences are valid only if the probability distribution used really explains the probabilistic behavior of the data. The financial theory is mostly based on the assumption of normality; however, there exists empirical evidence which points out that random variables such as economic growth rates, stock returns and different stock market indexes of the world, among others, do not follow a normal distribution ([13, 16]). Frequency histograms, which are constructed with observations of these random variables, indicate that the probability distributions of these variables are asymmetric and/or with tails heavier than the normal distribution tails.

In the economic and financial fields it is important to identify the probability distribution of such variables as in these areas it is very important to make accurate predictions, that is, with the least possible uncertainty. Therefore, many authors have tried to identify valid probability models for the returns of the big stock markets

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worldwide ([15, 14, 2]).

The Mexican $S\&P/BMV\ IPC$ Index measures the efficiency of the stocks with greater economic liquidity listed in the Mexican Stock Exchange. It provides a wide and representative index which is easily replicable and covers the Mexican stock market. A number of authors have studied this index by considering different approaches such as the study of extreme variations based on the Pareto-Levy distribution ([8]), and the dynamic volatility movements and market risk of the high-frequency Mexican IPC ([7]).

Here, we propose the use of the skew-t distribution ([5]) for modeling the probability behavior of the returns of representative indexes of exchange stock markets. In particular, we are interested in modeling the returns registered in time periods with important economic variations such as periods of new governments, crisis or events that have had a significative impact in the Mexican economy. The economic return is defined as the profit after a certain period of time, and it is calculated as the ratio $r = (final\ price - initial\ price)/(initial\ price)$.

Two data sets containing the weekly returns of the Mexican $S\&P/BMV\ IPC$ Index registered in two important economic periods are considered here. The first data set (see Table 3 and Figure 2) contains the weekly returns registered from January 2^{nd}, 2008 to December 31^{st}, 2009, which was a period of financial crisis at an international level that directly affected the Mexican economy. The second data set contains the weekly returns of the Mexican $S\&P/BMV\ IPC$ Index from December 2^{nd}, 2018 to July 12^{th} (see Table 6 and Figure 5). This period of time includes both the beginning of President Andrés Manuel López Obrador's rule and part of the global sanitary crisis because of the COVID-19 pandemic.

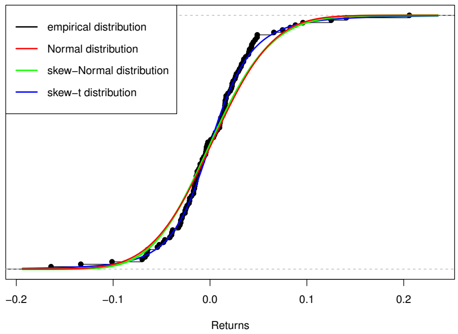
Figure 1 presents the empirical distribution function (ECDF) and three different fitted probability distributions to these data sets: the normal, skew-normal and skew-t distribution. It is observed that the normal and skew-normal distributions have a lack of fit in both cases; meanwhile, the skew-t distribution seems to provide a good fit. Hence, in order to formally test if the skew-t distribution provides a good fit to a data set, here we introduce a new property of this family of distributions and use it to propose a procedure for testing goodness of fit of this model.

1.1. The skew-t distribution

The skew-normal distribution ([3]) is a continuous probability distribution that extends and generalizes the normal distribution by using an additional shape parameter that regulates the asymmetry. Let φ and Φ be the probability density function (pdf) and the cumulative distribution function (cdf) of a standard normal random variable, respectively. It is said that the random variable Z_0 follows a skew-normal distribution with slant parameter $\alpha \in \mathbb{R}$, denoted as $Z_0 \sim SN(\alpha)$, if its pdf is given by:

$$f_{Z_0}(z;\alpha) = 2\varphi(z)\Phi(\alpha z), \quad -\infty < z < \infty. \tag{1}$$

Let $\xi \in \mathbb{R}$ and $\omega \in \mathbb{R}^+$, if $Z_0 \sim SN(\alpha)$ then the random variable $Y_0 = \xi + \omega Z_0$ is said to have a skew-normal distribution with location, scale and slant parameters ξ, ω and



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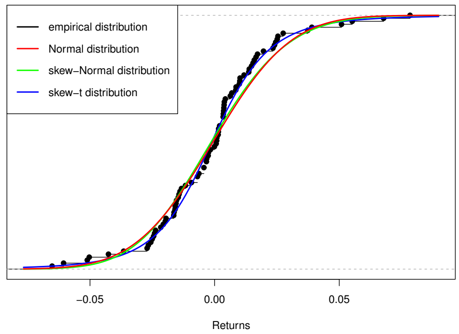


b

Figure 1. Fitted distributions to the weekly $S\&P/BMV\ IPC$ returns (a) from January 2^{nd}, 2008 to December 31^{st}, 2009 and (b) from December 2^{nd}, 2018 to July 12^{th}, 2020.

α, respectively, and it is denoted as $Y_0 \sim SN(\xi, \omega^2, \alpha)$.

The skew-t family of distributions ([5]) emerges from the skew-normal distribution. The main characteristic of this family is its flexibility for modeling asymmetric data which have heavier tails than those of the normal distribution. The standard Cauchy distribution, the Student's t-distribution, the normal and skew-normal distributions are included within this family of distributions.

Let $Z = \frac{Z_0}{\sqrt{V}}$ be, where $Z_0 \sim SN(\alpha)$ and $V \sim \chi^2_\nu/\nu$ are independent random variables, χ^2_ν denotes the Chi-square distribution with ν degrees of freedom. The random variable Z follows a skew-t distribution with slant parameter α and ν degrees of freedom and its pdf is given by:

$$t(z;\alpha,\nu) = 2t(z;\nu)T\left(\alpha z\sqrt{\frac{\nu+1}{\nu+z^2}};\nu+1\right),\ \ z,\alpha\in\mathbb{R},\ \ \nu>0, \tag{2}$$

where $t(z;\nu)$ represents the pdf of the Student-t distribution with ν degrees of freedom and $T(*;\nu+1)$ denotes the cdf of the Student's t distribution with $\nu+1$ degrees of freedom.

Some properties of this density are:

(1) If $\alpha = 0$, equation (2) reduces to the pdf of the Student's t distribution with ν degrees of freedom.
(2) If $\nu \to \infty$, equation (2) converges to the pdf of the $SN(\alpha)$ distribution.
(3) The random variable $Z^2 \sim F(1,\nu)$, where $F(\nu_1,\nu_2)$ denotes the Snedecor distribution with ν_1 and ν_2 degrees of freedom.

The family of distributions with pdf given in (2) can be extended by including a location parameter $\xi \in \mathbb{R}$ and a scale parameter $\omega \in \mathbb{R}^+$, considering the transformation $Y = \xi + \omega Z$. The random variable Y is said to have a skew-t distribution with parameters $(\xi, \omega^2, \alpha, \nu)$ and is denoted as $Y \sim ST(\xi, \omega^2, \alpha, \nu)$.

Let

$$b_\nu = \frac{\sqrt{\nu}\Gamma\left(\frac{1}{2}(\nu-1)\right)}{\sqrt{\pi}\Gamma\left(\frac{1}{2}\nu\right)}, \quad \nu > 1,$$
$$\text{and} \quad \delta = \frac{\alpha}{\sqrt{1+\alpha^2}}, \qquad \delta \in (-1, 1).$$

The first moments of Y are:

$$\mu = \mathbb{E}\left\{Y\right\} = \xi + \omega b_\nu \delta, \quad \nu > 1, \tag{3}$$

$$\sigma^2 = var\left\{Y\right\} = \omega^2 \left[\frac{\nu}{\nu-2} - (b_\nu\delta)^2\right] = \omega^2\sigma_z^2, \quad \nu > 2, \tag{4}$$

$$\gamma_1 = \frac{b_\nu\delta}{\sigma_z^{3/2}}\left[\frac{\nu(3-\delta^2)}{\nu-3} - \frac{3\nu}{\nu-2} + 2(b_\nu\delta)^2\right], \quad \nu > 3, \tag{5}$$

$$\gamma_2 = \frac{1}{\sigma_z^4}\left[\frac{3\nu^2}{(\nu-2)(\nu-4)} - \frac{4(b_\nu\delta^2)\nu(3-\delta^2)}{\nu-3} + \frac{6(b_\nu\delta)^2\nu}{\nu-2} - 3(b_\nu\delta)^4\right] - 3, \quad \nu > 4. \tag{6}$$

The coefficients γ_1 and γ_2 represent the third and fourth standardized cumulants of Y, respectively. The range of γ_1 is $(-4, 4)$ when $\nu > 4$. For $\nu \leq 3$, γ_1 does not exist. At least one of the tails of the distribution gets heavier when $\nu \to 0$, given the connection with the Cauchy distribution and the Student's t- distribution. If $\nu \to 4^+$ then the range of γ_2 is $[0, \ \infty)$.

This article is organized as follows. A goodness of fit test for the skew-t distribution hypothesis is introduced in Section 2. The results of a Monte Carlo simulation study on the power properties of the proposed methods for testing the skew-t distribution are presented in Section 3. In Section 4, the data sets containing weekly returns of the Mexican *S&P/BMV IPC* Index are further analyzed. Finally, in Section 5 some conclusions are established.

2. Methodology

This section presents a statistical test for the problem of testing the skew-t distribution hypothesis based on a random sample.

2.1. A new property of skew-t distributions

Consider the following result.

Theorem 2.1. *Let Z be a random variable such that $Z \sim ST(0, 1, \ \alpha, \ \nu)$, then the* *random variable Y^0, defined as*

$$Y^0 = \begin{cases} Z & \textit{with probability } 1/2, \\ -Z & \textit{with probability } 1/2, \end{cases} \tag{7}$$

follows a Student's t distribution with ν degrees of freedom.

Proof. Y^0 is a mixture of two random variables, then its pdf is given by

$$f_{Y^0}(y) = \frac{1}{2} f_Z(y) + \frac{1}{2} f_{-Z}(y). \tag{8}$$

Notice that the cdf of $-Z$ is $F_{-Z}(z) = 1 - F_Z(-z)$. Then $f_{-Z}(z) = \frac{d}{dz} F_{-Z}(z) = f_Z(-z)$. Hence, by (2),

$$f_Z(-z) = 2t(z;\nu) T\left((-\alpha) z \sqrt{\frac{\nu+1}{\nu+z^2}}; \nu+1\right), \tag{9}$$

since the Student's t distribution is a symmetric function. From equation (9) it follows that $-Z \sim ST(0, 1,\ -\alpha,\ \nu)$. Therefore, the pdf of the random variable Y^0 is given by:

$$f_{Y^0}(y) = t(y;\nu)\left[T\left(\alpha y \sqrt{\frac{\nu+1}{\nu+y^2}}; \nu+1\right) + T\left((-\alpha) y \sqrt{\frac{\nu+1}{\nu+y^2}}; \nu+1\right)\right].$$

Now, notice that $T(-a;\nu) = 1 - T(a;\nu)$, $a \in \mathbb{R}$, where T denotes the cdf of the standard Student's t distribution with ν degrees of freedom. Therefore,

$$\begin{aligned} f_{Y^0}(y) &= t(y;\nu)\left[T\left(\alpha y \sqrt{\frac{\nu+1}{\nu+y^2}}; \nu+1\right) + 1 - T\left(\alpha y \sqrt{\frac{\nu+1}{\nu+y^2}}; \nu+1\right)\right] \\ &= t(y;\nu), \end{aligned} \tag{10}$$

that is, Y^0 follows a standard Student's t distribution. □

If $Y \sim ST(\xi, \omega^2, \alpha, \nu)$, by the previous Theorem, the random variable Y^0 defined in (7) follows a standard Student's t distribution with $Z = (Y - \xi)/\omega$.

2.2. A test for skew-t distributions based on a data transformation

Let $Y_1, \ldots, Y_n$ be a random sample of size n from the $ST(\xi, \omega^2, \alpha, \nu)$ distribution. If the parameters ξ and ω are replaced by consistent estimators $\hat{\xi}$ and $\hat{\omega}$, then by the above result, the random variables

$$Y_i^0 = \begin{cases} Z_i' & \textit{with probability } 1/2, \\ -Z_i' & \textit{with probability } 1/2, \end{cases} \tag{11}$$

follow approximately a Student's t distribution for large sample sizes, where

$$Z_i' = \frac{Y_i - \hat{\xi}}{\hat{\omega}}, \quad i = 1, \ldots, n. \tag{12}$$

Let $X_1, \ldots, X_n$ be a random sample of size n coming from a continuous population. For testing the null hypothesis

$$H_0 : X_1 \ldots, X_n \sim ST(\xi, \omega^2, \alpha, \nu), \quad \text{with unknown values of } \xi, \omega^2, \alpha, \nu, \tag{13}$$

versus H_1 : not H_0, for large sample sizes we propose to use Anderson-Darling test ([1]) for testing the null hypothesis

$$H_0' : X_1', \ldots, X_n' \sim \text{ Student's t distribution}, \tag{14}$$

where

$$X_i' = \begin{cases} \frac{X_i - \hat{\xi}}{\hat{\omega}} & \textit{with probability } 1/2, \\ -\frac{X_i - \hat{\xi}}{\hat{\omega}} & \textit{with probability } 1/2, \end{cases} \quad i = 1, \ldots, n. \tag{15}$$

Anderson-Darling test rejects H_0' at a test size $\gamma \in (0, 1)$ if statistic:

$$A_n^2 = n \int_{-\infty}^{\infty} \frac{[T(x) - F_n(x)]^2}{T(x)[1 - T(x)]} dT(x) \tag{16}$$

is larger than $A^2_{1-\gamma}$, where T is the cdf of the Student's t distribution, $F_n(y)$ is the ECDF of $X_1', \ldots, X_n'$ and the cutoff point $A^2_{1-\gamma}$ is the $100(1-\gamma)\%$ quantile of the null distribution of A_n^2. The distribution of A^2 under H_0 depends on the hypothesized distribution, then $A^2_{1-\gamma}$ can be obtained by parametric bootstrap.

If H_0' is rejected at a test size $\gamma \in (0, 1)$ then H_0 in (13) is rejected at the same test size.

Remark. A bootstrap version of Anderson-Darling test is also considered here for testing (13). The test statistic compares the ECDF of $X_1, \ldots, X_n$ and the fitted skew-t distribution to $X_1, \ldots, X_n$.

3. Results and Discussion

In order to estimate the nominal test size and power of the proposed test ($W^\star$) and the Anderson-Darling test (W), a Monte Carlo simulation study was performed under the following conditions. Samples of sizes n = 50, 100, 500 were simulated and the nominal test size was fixed at $\alpha^\star = 0.05$. The size and power of the W and $W^\star$ tests were estimated by using 1000 Monte Carlo samples. In all cases, 200 bootstrap samples were simulated. The numbers of Monte Carlo and bootstrap samples considered are small because the algorithms used in this study are computationally expensive. The calculations were done in R ([17]). The sn package ([4]) of R was used to compute parameter estimates for the skew-normal and skew-t distributions.

For the estimation of the test sizes, different values were used for the slant parameter $\alpha \in [-20, 20]$ and the degrees of freedom $\nu > 1$. The values of the location and scale parameters were set at 0 and 1 because the two tests are invariant under changes in these parameters, and when these parameters are estimated by using equivariant estimators of location and scale (as are the maximum likelihood estimators), the sampling distributions of the statistics of the empirical distribution function do not depend on the real values of these parameters ([9]).

Table 1. Estimated size of the tests for different values of α and $\nu = 3, 5, 8, 12$ with $\alpha^{\star} = 0.05$.

α	ν	n=50		n=100		n=500	
		W	$W^{\star}$	W	$W^{\star}$	W	$W^{\star}$
20	3	0.03	0.02	0.2	0.2	0.06	0.03
15	3	0.02	0.02	0.02	0.02	0.05	0.05
10	3	0.02	0.01	0.02	0.02	0.06	0.04
5	3	0.01	0.04	0.03	0.03	0.05	0.04
2	3	0.01	0.02	0.04	0.03	0.06	0.04
0	3	0.02	0.02	0.05	0.04	0.06	0.04
-2	3	0.01	0.03	0.03	0.03	0.06	0.02
-5	3	0.01	0.02	0.03	0.02	0.05	0.04
-10	3	0.02	0.02	0.02	0.02	0.05	0.05
-20	3	0.03	0.03	0.02	0.02	0.05	0.04
20	5	0.03	0.03	0.02	0.02	0.06	0.04
15	5	0.03	0.02	0.03	0.02	0.04	0.04
10	5	0.01	0.02	0.02	0.02	0.05	0.04
5	5	0.01	0.03	0.02	0.02	0.05	0.03
2	5	0.02	0.02	0.03	0.03	0.05	0.03
0	5	0.03	0.03	0.04	0.03	0.06	0.03
-2	5	0.01	0.03	0.04	0.03	0.07	0.03
-5	5	0.01	0.03	0.02	0.03	0.04	0.03
-10	5	0.02	0.03	0.02	0.03	0.04	0.04
-20	5	0.03	0.03	0.03	0.03	0.08	0.05
20	8	0.03	0.03	0.04	0.03	0.05	0.04
15	8	0.02	0.03	0.04	0.03	0.05	0.03
10	8	0.01	0.03	0.03	0.03	0.08	0.04
5	8	0.01	0.02	0.03	0.04	0.04	0.03
2	8	0.02	0.03	0.04	0.03	0.06	0.03
0	8	0.03	0.03	0.04	0.03	0.05	0.03
-2	8	0.03	0.04	0.04	0.02	0.05	0.03
-5	8	0.02	0.03	0.02	0.03	0.05	0.02
-10	8	0.01	0.03	0.03	0.03	0.04	0.03
-20	8	0.03	0.03	0.05	0.03	0.08	0.03
20	12	0.04	0.03	0.05	0.03	0.05	0.03
15	12	0.02	0.03	0.04	0.04	0.03	0.04
10	12	0.01	0.03	0.02	0.03	0.05	0.03
5	12	0.01	0.04	0.02	0.02	0.05	0.04
2	12	0.02	0.03	0.04	0.03	0.05	0.03
0	12	0.02	0.03	0.04	0.03	0.06	0.03
-2	12	0.02	0.04	0.04	0.03	0.05	0.03
-5	12	0.01	0.04	0.02	0.03	0.05	0.03
-10	12	0.01	0.02	0.04	0.04	0.05	0.02
-20	12	0.03	0.03	0.04	0.03	0.10	0.03

The powers of the W and $W^{\star}$ tests were estimated under the following alternative distributions.

(1) N(0, 1): standard normal.
(2) Cauchy(0, 1): standard Cauchy.
(3) Logistic(0, 1): standard logistic.
(4) Beta(a, b): beta with parameters a and b.
(5) Uniform(0, 1).
(6) Laplace(0, 1): standard Laplace.
(7) Gamma(a, b): Gamma with shape and scale parameters a and b. When $a = 1$ and $b = 1$ the density of the standard exponential distribution is obtained: Exp (1). When $a = 4/2$ and $b = 2$ the chi-square distribution with 4 degrees of freedom is obtained, and it is denoted as χ^2_4.
(8) Gumbel(0, 1): standard Gumbel.

(9) sLaplace(0, 1, λ): standard asymmetric Laplace with slant parameter $\lambda \in \mathbb{R}$ and probability density function (pdf) given by:

$$f_X(x) = \begin{cases} \frac{\lambda e^{-\lambda x}}{1+\lambda^2} & x \geq 0, \\ \frac{\lambda e^{x/\lambda}}{1+\lambda^2} & x < 0. \end{cases}$$

(10) Log-normal(μ, σ): Log-normal with location parameter $\mu \in \mathbb{R}$ and scale parameter $\sigma > 0$.

(11) Weibull(λ, k): Weibull with scale parameter $\lambda > 0$ and shape parameter $k > 0$.

(12) GP(λ, 1): generalized Pareto distribution with shape parameter $\lambda \in \mathbb{R}$ and pdf given by:

$$f_X(x) = (1 + \lambda x)^{-(1/\lambda+1)},$$

where $x \geq 0$ for $\lambda \geq 0$ and $0 \leq x \leq -1$ for $\lambda < 0$.

(13) DGP(α): generalized Double Pareto. The following result was used to generate random numbers from this distribution.

Let X and Y be independent random variables with GP(α, 1) distribution, then the variable $Z = X - Y$ has a generalized Double Pareto distribution with parameter α.

Note that in the list of distributions the N(0, 1) and Cauchy (0, 1) distributions were included, but they belong to the family of skew-t distributions for certain values of the skew-t parameters.

Table 2. Estimated power of the tests with $\alpha^\star = 0.05$.

Alternative	γ_1	γ_2	n=50		n=100		n=500	
			W	$W^\star$	W	$W^\star$	W	$W^\star$
			Symmetric distributions					
Beta(0.5, 0.5)	0	-1.5	0.91	0.60	1	0.99	1	1
N(0, 1)	0	0	0.02	0.04	0.06	0.04	0.07	0.03
Cauchy(0, 1)	$\star$	$\star$	0.02	0.05	0.02	0.07	0.01	0.04
Logistic(0, 1)	0	1.2	0.02	0.04	0.05	0.04	0.05	0.05
Uniform(0, 1)	0	1.8	0.28	0.18	0.96	0.78	1	1
Laplace(0, 1)	0	3	0.05	0.05	0.12	0.07	0.63	0.33
			Skew distributions					
sLaplace(0, 1, 3)	-1.96	5.8	$\star$	0.29	$\star$	0.80	$\star$	1
sLaplace(0, 1, 1.5)	-1.4	4.34	0.07	0.06	0.30	0.07	0.99	0.29
Beta(1, 3)	0.75	0.10	0.15	0.04	0.17	0.08	0.17	0.21
Gamma(4, 1)	1	1.5	0.01	0.03	0.04	0.04	0.15	0.04
Gumbel(0, 1)	1.14	2.4	0.01	0.04	0.03	0.05	0.09	0.04
sLaplace(0, 1, 0.7)	1.27	4.12	0.06	0.05	0.22	0.07	0.95	0.21
χ^2_4	1.41	3	0.01	0.04	0.02	0.04	0.16	0.08
Log-normal(0, 0.5)	1.75	5.9	0.01	0.03	0.02	0.03	0.10	0.05
sLaplace(0, 1, 0.5)	1.8	5.33	0.24	0.09	0.84	0.17	1	1
Exp(1)	2	6	0.18	0.06	0.25	0.08	0.63	0.39
Gamma(0.5, 1)	2.83	12	$\star$	0.13	$\star$	0.25	$\star$	1
Weibull(0.75, 1)	3.1	16.03	$\star$	0.10	0.54	0.17	0.07	0.97
GP(0.15, 1)	3.5	30.5	0.17	0.06	0.26	0.09	0.69	0.28
Log-normal(0, 1.5)	33.47	10075.25	$\star$	0.07	$\star$	0.07	$\star$	0.18
GP(0.5, 1)	$\star$	$\star$	$\star$	0.07	$\star$	0.07	$\star$	0.15
DGP(0.4)	$\star$	$\star$	0.04	0.06	0.08	0.08	$\star$	0.17
DGP(2.5)	$\star$	$\star$	$\star$	0.11	$\star$	0.20	$\star$	0.41

$\star$ Undefined values.

Table 1 presents the estimated test sizes for the W and $W^\star$. In general, both tests preserve the nominal size for samples of sizes n=50 and 100, but for samples of size

Table 3. Weekly *S&P/BMV IPC* returns from January 2^{nd}, 2008 to December 31^{st}, 2009.

-0.0132	0.0227	-0.0021	-0.0063	0.0432	0.0036	-0.0143	0.0369	0.0427	-0.0644
0.0036	0.0229	0.0467	-0.0028	-0.0393	0.0167	0.0402	-0.0102	0.0104	0.0165
-0.0115	0.0408	0.0149	0.0345	0.0882	-0.0162	-0.0170	0.0076	-0.0466	0.0220
0.0239	0.0098	0.0320	-0.0303	0.0959	-0.0301	0.0157	0.0828	-0.0190	0.0305
0.0488	-0.0038	0.1407	-0.0397	-0.0312	-0.0543	-0.0523	0.0447	0.0112	-0.0480
0.0651	-0.0649	0.0327	0.0132	0.0396	0.0659	-0.0211	0.1251	-0.0670	-0.0153
-0.0284	0.2058	-0.1641	0.0154	-0.1334	-0.1012	-0.0042	0.0107	-0.0142	-0.0158
-0.0217	-0.0170	0.0099	0.0057	-0.0047	-0.0385	0.0186	-0.0259	-0.0327	-0.0081
0.0289	-0.0236	-0.0256	0.0292	-0.0132	0.0265	0.0046	-0.0148	-0.0247	0.0158
-0.0079	0.0486	0.0349	0.0008	0.0153	-0.0101	-0.0209	0.0273	0.0208	-0.0415
0.0749	0.0257	-0.0700	0.0140						

n=500 it is observed that the size estimated with the W test sometimes exceeds the nominal size, especially in cases where the slant parameter is large and/or the degrees of freedom are greater than 3. On the other hand, the test $W^\star$, in general, has a good control of the type I error probability when the sample size is n = 500.

Table 2 presents the estimated powers of the tests for the different alternative distributions considered, which were classified as symmetric or asymmetric and ordered according to the values of their skewness (γ_1) and excess kurtosis (γ_2) coefficients. It is observed that both tests under study have high power for cases where the alternative distributions have a bounded support. For example, the power increases as the sample size increases under the Beta(0.5, 0.5) distribution and the Uniform(0, 1) distribution.

For samples of size n=100, the W test has high power against the asymmetric Laplace distribution and in general versus alternative distributions with coefficient $\gamma_2 < 30$. A similar observation is made for the $W^\star$ test; however, in this case the power increase is not very large, compared to the case when n=50.

For samples of size 500, the W test is powerful versus most alternative distributions considered. It is rather sensitive against the Laplace distributions, both symmetric and asymmetric, or related distributions such as the exponential distribution.

4. Applications

In this section we analyze the data sets discussed in the Introduction section.

In order to verify the assumption of independence among the observations, the Box-Pierce test ([6]) was applied by using the *Box.test* function of the *stats* package ([18]) of R ([17]).

The following probability distributions were fitted to each data set. The skew-t $ST(\xi, \omega^2, \alpha, \nu)$, the normal $N(\mu, \sigma^2)$, the skew-normal $SN(\xi, \omega^2, \alpha)$ and the Laplace $Lap(\mu, b)$. Parameter estimates of the skew-t and skew-normal distributions were calculated with the functions *st.mpl* and *sn.mpl* of the *sn* package ([4]) of R ([17]).

Different goodness-of-fit tests were applied to test each of the fitted probability distributions. The Shapiro-Wilk test was used for testing normality. The bootstrap

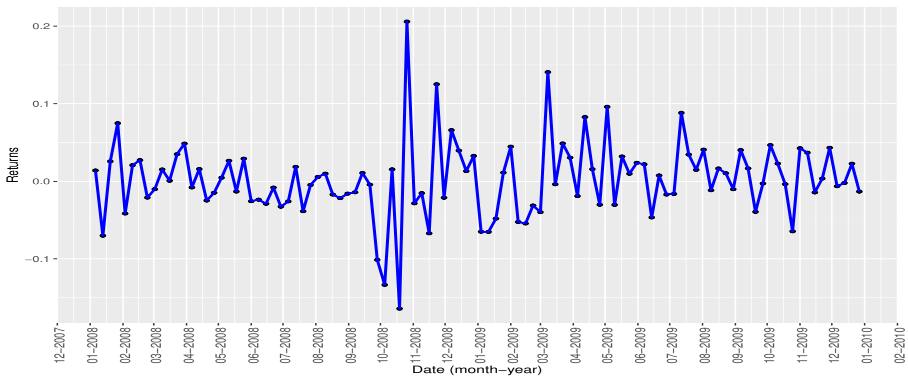


Figure 2. Weekly $S\&P/BMV\ IPC$ returns from January 2^{nd}, 2008 to December 31^{st}, 2009.

Table 4. Estimated parameters of different distributions fitted to the weekly $S\&P/BMV\ IPC$ returns from January 2^{nd}, 2008 to December 31^{st}, 2009.

Distribution	Estimates
$ST(\xi, \omega^2, \alpha, \nu)$	(-0.0047, 0.0322^2, 0.2063, 3.1754)
$N(\mu, \sigma^2)$	(0.0023, 0.0490^2)
$SN(\xi, \omega^2, \alpha)$	(-0.0347, 0.0614^2, 1.1852)
$Lap(\mu, b)$	(-0.0006, 0.0346)

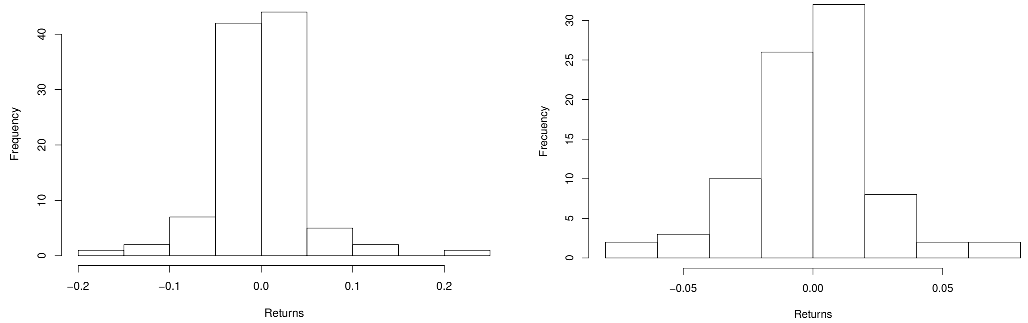
Anderson-Darling test, studied by González-Estrada and Cosmes-Martínez ([10]), was used to test the skew-normal distribution. The Anderson-Darling test, proposed by González-Estrada and Villaseñor A. ([11]) implemented in the function *laplace_test* of the *goft* package ([12]), was used to test the Laplace distribution. The tests proposed in Section 2 were applied to test the skew-t distribution. The R script for implementing these tests is provided in the Appendix. For these two tests and for the test for the skew-normal distribution, 500 bootstrap samples were used to estimate the distribution of the statistic and calculate the probability value.

For practical purposes, the used tests were denoted as follows:

BP: Box-Pierce test.
SW: Shapiro-Wilk test.
AD_N: Anderson-Darling test for the skew-normal distribution.
AD_{LP}: Anderson-Darling test for the Laplace distribution.
W: Anderson-Darling test for the skew-t distribution.
$W^{\star}$: Anderson-Darling test for the skew-t distribution with transformed data.

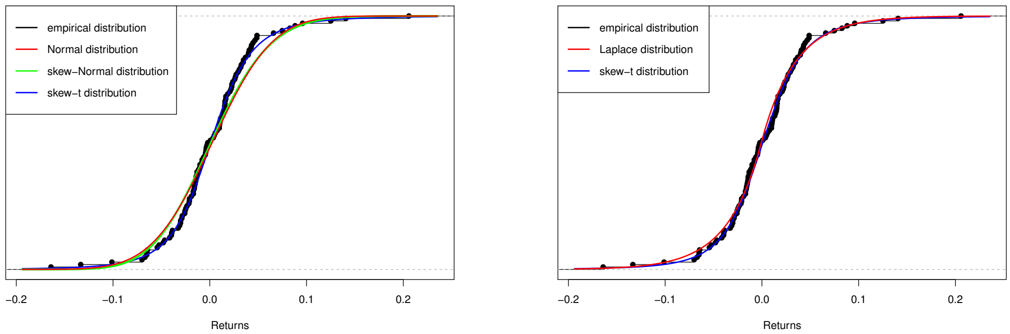
Figure 3(a) presents a frequency histogram of the returns from January 2^{nd}, 2008 to December 31^{th}, 2009. Note that the distribution is leptokurtic with a slight elongation on the right tail. The estimated skewness and excess kurtosis coefficients are 0.4364 and 3.5761, respectively.

Table 5 presents the results of the independence and goodness of fit tests applied to this data set. Notice that the distribution of the data is neither normal nor skew-normal since the corresponding null hypotheses were rejected by using the SW and AD_N tests. The lack of fit with these distributions is observed in Figure 4(a). On the other hand, the skew-t distribution hypothesis is not to rejected, which agrees



a b

Figure 3. Frequency histograms of (a) the weekly returns from January 2^{nd}, 2008 to December 31^{st}, 2009 and (b) the weekly returns from December 2^{nd}, 2018 to July 12^{th}, 2020.



a b

Figure 4. Fitted distributions to the weekly *S&P/BMV IPC* returns from January 2^{nd}, 2008 to December 31^{st}, 2009.

with the evidence provided by Figure 4(a), where it is observed that the fitted skew-t distribution is close to the empirical distribution function (ECDF) of the observations. Note that the Anderson-Darling test for the Laplace distribution rejects the corresponding null hypothesis. The fitted Laplace distribution is shown in Figure 4(b).

The second data set includes the weekly *S&P/BMV IPC* returns from December 2^{nd}, 2018 to July 12^{th}, 2020, with a total of 85 observations. The estimated skewness and excess kurtosis coefficients are 0.3044 and 1.4889, respectively. The corresponding frequency histogram is shown in Figure 3(b).

Table 5. Results of the different tests applied to the weekly *S&P/BMV IPC* returns from January 2^{nd}, 2008 to December 31^{st}, 2009.

Test	BP	SW	AD_N	AD_{LP}	W	$W^\star$
Statistic	3.3302	0.9366	1.4081	1.6155	0.1965	0.4001
p-value	0.0680	8.771e-05	0	0.0214	0.618	0.228

Table 6. Weekly $S\&P/BMV\ IPC$ returns from December 2^{nd}, 2018 to July 12^{th}, 2020.

0.0034	-0.0426	0.0085	-0.0245	0.0254	-0.0365	0.0784	0.0086	-0.0017	-0.0511
0.0374	0.0509	0.0006	0.0014	0.0275	-0.0266	-0.0198	-0.0652	-0.0503	-0.0001
0.0606	-0.0043	0.0137	0.0015	-0.0155	-0.0146	0.0245	0.0038	0.0069	-0.0069
0.0037	0.0552	-0.0216	-0.0153	0.0036	-0.0033	-0.0026	0.0099	0.0035	0.0016
0.0043	0.0148	-0.0133	0.0172	0.0008	0.0020	0.0678	0.0118	-0.0241	0.0154
-0.0164	-0.0251	-0.0256	-0.0195	0.0043	-0.0093	0.0102	-0.0069	0.0101	0.0034
-0.0206	0.0063	-0.0159	-0.0158	-0.0117	0.0196	-0.0063	0.0390	0.0234	-0.0033
-0.0139	-0.0237	-0.0271	0.0166	-0.0032	-0.0163	0.0034	-0.0143	0.0159	0.0251
0.0240	0.0009	0.0039	-0.0139	-0.0025					

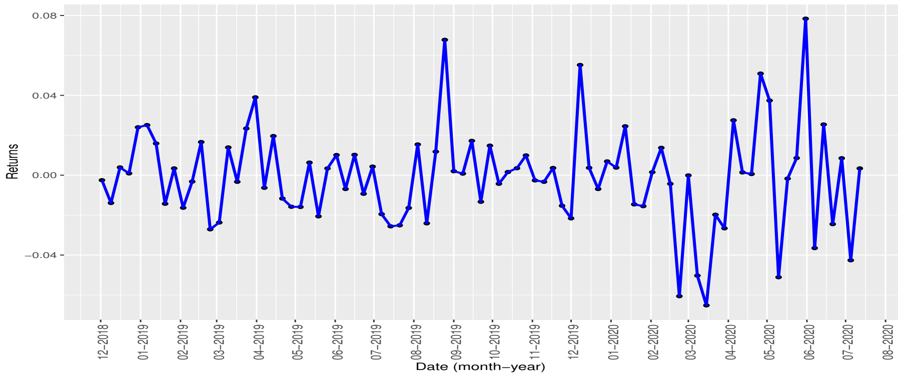
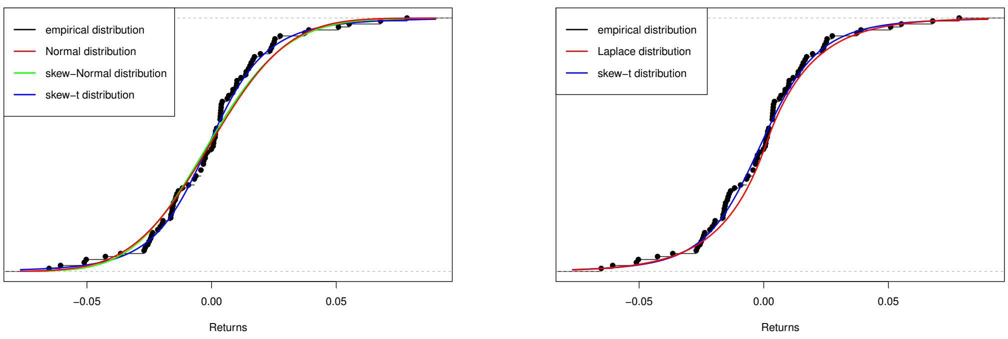


Figure 5. Weekly $S\&P/BMV\ IPC$ returns from December 2^{nd}, 2018 to July 12^{th}, 2020.

Table 7. Estimated parameters of different distributions fitted to the weekly $S\&P/BMV\ IPC$ returns from December 2^{nd}, 2018 to July 12^{th}, 2020.

Distribution	Estimates
$ST(\xi, \omega^2, \alpha, \nu)$	(-0.0027, 0.0175^2, 0.1040, 3.6134)
$N(\mu, \sigma^2)$	(-0.0007, 0.0244^2)
$SN(\xi, \omega^2, \alpha)$	(-0.0194, 0.0308^2, 1.1877)
$Lap(\mu, b)$	(8e-04, 0.0176)



a b

Figure 6. Fitted distributions to the weekly $S\&P/BMV\ IPC$ returns from December 2^{nd}, 2018 to July 12^{th}, 2020.

Table 8. Results of the different tests applied to the weekly returns of the $S\&P/BMV\ IPC$ from December 2^{nd}, 2018 to July 12^{th}, 2020.

Test	BP	SW	AD_N	AD_{LP}	W	$W^\star$
Statistic	0.0371	0.9637	0.9136	0.5434	0.2854	0.1924
p-value	0.8471	0.0172	0.004	0.4529	0.238	0.8120

From the results presented in Table 8 and considering a test size equal to 0.05, we have that the independence hypothesis is not rejected by Box-Pierce test. Both the normal and skew-normal distribution hypotheses are rejected, which agrees with the evidence provided in Figure 6(a), which depicts the ECDF of the observations and the fitted normal and skew-normal and skew-t distributions. It is observed that the fitted normal and skew-normal distributions are very similar, but the skew-t distribution provides a better fit around the center of the data set. On the other hand, the skew-t distribution hypothesis is not rejected with the W and $W^\star$ tests. Note that even though the Laplace distribution hypothesis is not rejected, it is observed in Figure 6(b) that the skew-t distribution provides a better fit to this data than the Laplace distribution.

5. Conclusions

In this article, the use of the skew-t probability distribution has been proposed for modeling the returns of the $S\&P/BMV\ IPC$ Index. In order to formally assess if this model provides a good fit to real data sets containing a realization of a random sample, two goodness-of-fit tests (W and W^*) have been proposed for the skew-t distribution hypothesis when parameters are unknown. The analysis of two real data sets containing weekly returns of the Mexican Exchange Market Index, lead us to consider the skew-t distribution as a plausible probability model for such variable. Monte Carlo simulation results indicate that both tests in general preserve the nominal test size under the studied conditions. The W test is in general more powerful than the $W^\star$ test against the considered alternative distributions.

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Disclosure statement

No potential conflict of interest was reported by the authors.

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Appendix A. Anderson-Darling test for skew-t distributions

```
#W test
install.packages("sn")
library(sn)
#Anderson-Darling statistic
AD_stat2<- function(x){
  n<- length(x) # sample size
  w<- st.mple(y = x, opt.method = "nlminb")$dp
  s<- sort(x)
```

```
  theop<-  pst(s, w[1], w[2], w[3],w[4])
  ad_calc<- - n - sum((2*(1:n)-1)*log(theop) +
  (2*n+1-2*(1:n))*log(1-theop))/n
  return(ad_calc)
}
# The following function returns 1 if Ho is rejected
# and 0 otherwise.
# Arguments
# x: data (vector)
# B: number of bootstrap samples
# size: test size

AD_test2 <- function(x, B, size)
{
  n<- length(x)
  ad_calc <- AD_stat2(x)
  w    <- st.mple(y=x,opt.method="nlminb")$dp
  #bootstrap
  AD_dist <- replicate(B, AD_stat2(rst(n, xi = w[1],
  omega=w[2], alpha = w[3],nu=w[4])))
  orden<- sort(AD_dist)
  cuant<- B*(1-size)
  k_alpha <- orden[cuant]
  return(ifelse(ad_calc > k_alpha, 1, 0))
}
```

Appendix B. Test for skew-t distributions based on transformed data

```
#W* test
install.packages("sn")
library(sn)
#Anderson-Darling statistic
ADstat <- function(x){
  n<- length(x) # sample size
  w<- st.mple(y=x,opt.method="nlminb",symmetr = TRUE)$dp
  s <- sort(x)
  theop <-  pst(s,w[1],w[2],0,w[3])
  ad_calc <- -n- sum((2*(1:n)-1)*log(theop)
  + (2*n+1-2*(1:n))*log(1-theop))/n
  return(ad_calc)
}
ADtest<- function(x, B, size)
{
  n<- length(x)
  ad_calc<- ADstat(x)
  w<- st.mple(y=x,opt.method="nlminb",symmetr = TRUE)$dp
  AD_dist <- replicate(B, ADstat(rst(n,w[1],w[2],0,w[3])))
  orden   <- sort(AD_dist)
  cuant   <- B*(1-size)
```

```
  k_alpha <- orden[cuant]
  return(ifelse(ad_calc > k_alpha, 1, 0))
}
#Anderson-Darling test based on a data transformation
# The following function returns 1 if Ho is rejected
# and 0 otherwise.
# Arguments
# x: data (vector)
# B: number of bootstrap samples
# size: test size

test<-function(x,B,size) {
  n<-length(x)
  estimation<-st.mple(y = x, opt.method = "nlminb")$dp
  x1<-(x-estimation[1])/estimation[2]
  #set.seed(1) use for p-value calculation
  A<-rbinom(n,1,0.5)
  D<-matrix(NA, ncol = 1, nrow = n)
  for(i in 1:n ){
    if (A[i]==1 ) {D[i]<-x1[i]}  else D[i]<--(x1[i])}
  return(ADtest(D,B,size))
}
```